The following homework contains proof fundamentals such as direct argument, contradiction, and induction, as well as limits and sequences. Feel free to work with each other. Please write your final submission on paper without lines. It is due on **Friday, January 17**.

Problem 1

An n-gon is a polygon with n sides. A polygon is **convex** provided all of its angles are less than 180 degrees. A **diagonal** of a convex polygon is a line segment that can be drawn between two vertices that are not adjacent. Prove that a convex n-gon has $\frac{n(n-3)}{2}$ diagonals, for $n \geq 3$.

Problem 1 Solution: We show this by induction. Let $n = 3$. A convex 3-gon, i.e. a triangle, has 0 diagonals (all vertices are adjacent), which equals $\frac{3(3-3)}{2}$. Let the result hold for all convex polygons with up to k sides. Take any convex $(k+1)$-gon, and pick any vertex v. Draw a diagonal between the two vertices, u and w, that are adjacent to v, constituting 1 diagonal. This diagonal induces a k-gon using all vertices excluding v. To check that the k-gon is convex, all angles other than those at u and w are less than 180 degrees, as they are from the original $(k+1)$-gon. The angles of the k-gon at u and w are less than the angles of the $(k+1)$-gon at u and w, which are both less than 180 degrees. Given that the k-gon is convex, by assumption, there are $\frac{k(k-3)}{2}$ more diagonals in side of it. Finally, we add the diagonals from v to the $k-2$ other non-adjacent vertices. Summing them up, we have

$$\begin{aligned}\frac{k(k-3}{2} + k - 2 + 1 &= \frac{k(k-3) + 2k - 2}{2} \\ &= \frac{k^2 - 3k + 2k - 2}{2} \\ &= \frac{k^2 - k - 2}{2} \\ &= \frac{(k+1)(k-2)}{2} \\ &= \frac{(k+1)((k+1)-3)}{2}\end{aligned}$$

The equation holds for the convex $(k+1)$-gon, and by induction, holds for all n.

For Problem 2 and 3, we need to define the **supremum**, denoted $\sup S$, of set $S \subseteq \mathbb{R}$ as the least upper bound of the set. Precisely, u is an **upper bound** for S if for all $x \in S$, $u \geq x$. A number $s = \sup S$ if it is an upper bound of S, and for any other upper bound u of S, $s \leq u$.

Similarly, the **infimum**, denoted $\inf S$, is the greatest lower bound of S. Neither of these need to exist if the set is unbounded. For example, $\sup \mathbb{Z}$ can be considered ∞, as there is no upper bound.

Problem 2

(a) Give an example of a set where the supremum and infimum are **not** members of the set.

(b) Let S be a set, and $s = \sup S$. Prove that for any $\epsilon > 0$, there exists $x \in S$ such that

$$s - \epsilon \leq x \leq s$$

What is the analogous condition for the infimum of S?

Problem 2 Solution: The set $(0, 1)$ has infimum 0 and supremum 1, both of which are not included. Because $s = \sup S$, it is a upper bound for S, so any $x \in S$ will satisfy $x \leq s$. For the other inequality, assume otherwise, that there is some $\epsilon > 0$ such that for all $x \in S$, $x < s - \epsilon$. Then, $s - \epsilon$ is an upper bound for S. However, $s - \epsilon < s$, but s should the least upper bound by definition. This is a contradiction, and our original assumption was false. It must be that for all $\epsilon > 0$, there is an $x \in S$ with $s - \epsilon \leq x$. The analogous condition for the infimum $i = \inf S$ is that for all $\epsilon > 0$, there is an $x \in S$ with $i \leq x \leq i + \epsilon$.

Problem 3

Let (x_n) be a bounded sequence in $\mathbb{R}$. Prove that if (x_n) is monotone, then it converges, in two parts.

(a) Let $S = \{x_n : n = 1, 2, ...\}$ be the set that contains all the elements of (x_n). Prove that if (x_n) is monotone non-decreasing (i.e. $x_1 \leq x_2 \leq x_3 ...$), then $x_n \to \sup S$. Similarly, show that if (x_n) is monotone non-increasing, then $x_n \to \inf S$ (use Problem 2).

(b) Use the previous part to prove the original theorem.

Problem 3 Solution: For the first part, given any ϵ, there must be an $x \in S$ such that $s - \epsilon \leq x$, where $s = \sup S < \infty$. Becuase x is an element of the sequence, call it x_N where N is its index. Because the sequence is monotone non-decreasing, we have $x_N \leq x_n$ for all $n \geq N$. Thus, for all $n \geq N$, we have (as ϵ is arbitrary),

$$s - \epsilon \leq x_N \leq x_n \leq s \leq s + \epsilon \iff |x_n - s| < \epsilon$$

Thus, (x_n) converges to s by definition. A similar argument show the convergence of a bounded, monotone non-increasing sequence to the infimum of the sequence's values. As for the second part, a bounded monotone sequence is either monoton non-decreasing or non-increasing, and converges in either case.

Problem 4 (optional)

Prove that the closed interval $[0, 1]$ is uncountable.

Problem 4 Solution: We show this by a variant of Cantor's Diagonal Argument. In this case, we can represent any element $x \in [0, 1]$ be the sequence of a decimal expansion. For example, $0 = 0.000...$, $\frac{1}{3} = 0.333...$, $1 = 0.999...$, $\frac{1}{e} = 0.367...$, etc. Assume for the sake of contradiction that $[0, 1]$ is countable. Laying these out in the same table, we can generate an element in $[0, 1]$ be taking letting the n-th digit be equal to that of the n-th row in the table, and then changing all the elements to another decimal. This new element show be included in $[0, 1]$, but differs with every element in at least one digit. We should have laid them all out in our original table, so our original assumption of countability must be false.

Tips

- For Problem 1, it is logically insufficient to start the inductive step with a k-gon, and **generate** a $(k+1)$-gon by adding a vertex. The result should hold for **any** convex $(k+1)$-gon, not just the one that you generated.

- Similarly, very few people addressed convexity. The starting $(k+1)$-gon must be convex, and the induced k-gon must also be. I drew an example in one of the homeworks where a $(k+1)$-gon generated by adding a vertex can fail to be convex.

- Problem 1 can also be done using a method called **combinatorial proof**, in which one counts the left and right side of the equation in different ways to yield the same answer.

- That which you can prove directly, try not to obfuscate using contradiction and contrapositive. For example, the inequality in Problem 2 states $s - \epsilon \leq x \leq s$. You have $x \leq s$ by definition, as s is an upper bound.

- For Problem 4, be specific as to which infinite sequence (in this case, the decimal digits) corresponds to each element of the set. Also be careful about how you would change the diagonal, as some responses said that they would "negate" the diagonal, and digits cannot be negated.

- Note that in Problem 3, just because a set S is bounded above by M and below by $-M$, does not mean that $\sup S = M$ and $\inf S = -M$. M is just an upper bound, not necessarily the **least** upper bound. Similarly, $-M$ is just a lower bound, not not necessarily the **greatest** lower bound.

- Another strategy for Problem 4 was to find a unique correspondence between (0, 1) and every element of $\mathbb{R}$, a known uncountable set. Specifically, for every $a \in (0,1)$, there is a $b \in \mathbb{R}$ to which is uniquely correspond. To use this strategy, we must be sure that $a \mapsto b$ is one-to-one, i.e. no two $a_1, a_2 \in (0,1)$ map to the same $b \in \mathbb{R}$, and onto, i.e. for every $b \in \mathbb{R}$, there is an $a \in (0,1)$ that produces it.