A Mathematical Derivation of the Discounted Cash Flow Valuation Model
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1 Introduction

The Discounted Cash Flow (DCF) model is an extremely common valuation method. The underlying theory of the model is usually hidden in most expositions. This note uses tools from calculus to derive each of the formulas used in the DCF, providing both the mathematical and financial context.

2 Interest and the Time Value of Money

Money is a quantity that has value in time, because money that is not being used now can invest and grow. \$10 now can become \$15 within a year, so sitting on \$10 for a year without investing actually decreases its inherent value. This difference in value can be represented as such:

$$FV = P(1+r)^t$$

This is called the **simple interest formula**, where P is the **present value** of some "liquid enough" asset that can be used to buy into an investment opportunity. The FV is its **future value**, r is its (annual) rate of growth or **interest rate**, and t is the number of years in the future. When money increases its value due to interest at a certain point in time, we say that it **compounds**.

For example, investing \$100 with a 5% interest rate for 2 years should yield \$110.25. The reason this is an exponential function is because every time the interest rate is applied (compounded), it is applied to a larger amount. The first year is 5% on \$100, so \$105. The second year is not just the previous amount plus \$5, but instead 5% of \$105.

$$\$105 + 0.05 \cdot \$105 = \$105(1+0.05) = \$105(1.05) = \$100(1.05)(1.05) = \$100(1.05)^2$$

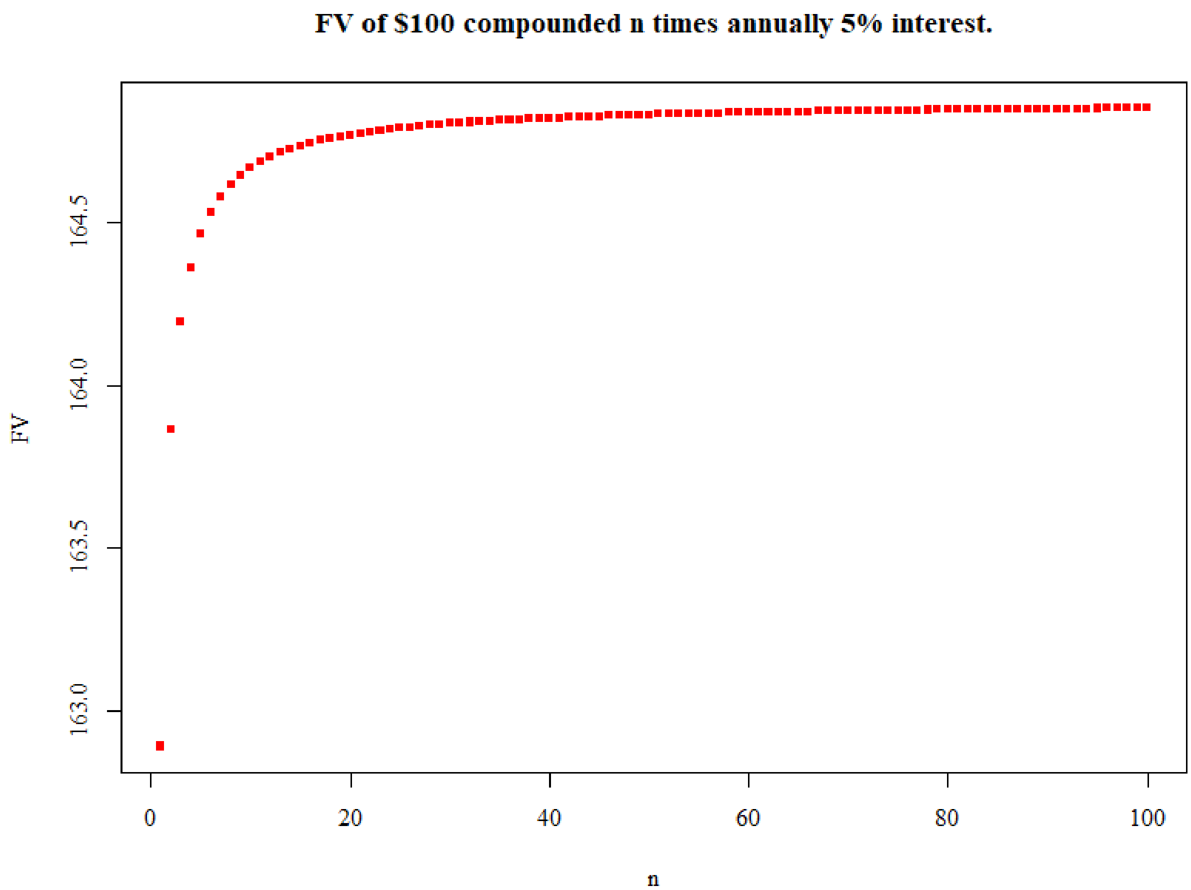
This is of course not the only compounding scheme possible for an interest rate. We could split the interest rate into four pieces, and compound each of them quarterly throughout the year. If this is the case, we can adjust the formula above as:

$$FV = P(1+\tfrac{r}{4})^{4t}$$

Here, we accrue a quarter of the rate each time, but 4 times as frequently. Compounding this way for 2 years gives us $\$100(1+\frac{0.05}{4})^8 = \$100(1.0125)^8 = \$110.49 > \$110.25$. Generally speaking, the future value of money compounded n times a year can be calculated as:

$$FV = P(1+\tfrac{r}{n})^{nt}$$

This called the **compound interest formula**. Compounding more frequently increased the future value of our principal amount, even though we let the same amount of time pass. Below, we plot the future value of \$100 compounded at 5% interest n times a year for 10 years, as a function of n.



Clearly, the future value plateaus - for a fixed amount P and fixed number of years t, compounding more frequently does not raise the future value of a principal amount to infinity. Rather, it pushes the FV to approach a certain limit. What is this limit? Recall that the function e^x (e being Euler's constant) has three common definitions:

$$e^x \approx 2.71828^x \tag{1}$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + ... \tag{2}$$

$$e^x = \lim_{n \to \infty} (1 + \tfrac{x}{n})^n \tag{3}$$

The first definition is an approximation of the constant itself, the second is the Taylor Series representation of the function, and the last is a limit definition. Using the last definition and the compound interest formula, we can figure out what the future value of an amount compounded continuously (at all points in time) is. To calculate this, we say the amount is compounded an "infinite" number of times a year, by taking the limit as n approaches infinity.

$$
\begin{aligned}
\lim_{n\to\infty} FV &= \lim_{n\to\infty} P(1+\tfrac{r}{n})^{nt} \\
&= P\left(\lim_{n\to\infty}(1+\tfrac{r}{n})^{nt}\right) \\
&= P\left[\left(\lim_{n\to\infty}(1+\tfrac{r}{n})^{n}\right)^{t}\right] \\
&= P(e^{r})^{t} \\
&= Pe^{rt}
\end{aligned}
$$

The third step follows from the fact the multiplication of exponents is the same as raising an expression with one exponent to the power of the other exponent. Using the same values from the graph above, we have $\$100(e^{0.05\cdot 10}) = \$164.87$. This is the line that the future value is approaching. $FV = Pe^{rt}$ is therefore known as the **continuously compounded interest formula**.

3 Weighted Average Cost of Capital (WACC) and Discounted Cash Flow (DCF)

3.1 WACC

We now introduce particular financial metrics. The first metric we introduce is the WACC - or weighted average cost of capital. This percentage represents the **required return** that investors should expect from the investment opportunity given by this company, in order to make up for the money spent on capital. We will denote it with w, and it is calculated as follows:

$$w = \frac{E}{E+D}\cdot R_E + \frac{D}{E+D}\cdot R_D \cdot (1 - r_{\text{tax}}),$$

where E and D are the equity and debt that the company has financed respectively, R_E and R_D are the cost of equity and cost of debt, and r_{tax} is the corporate tax rate. Companies that have low WACC are seen as more valuable and less risky, as there is a smaller required return for you to make up the money you spent on the company's capital. These costs are averaged (or weighted) by the proportion of equity financing and deb financing the company uses. If the company sells more debt than equity, then the cost of debt term will have a larger affect on the WACC. In the next section, you will see the application of WACC in discounting the value of an investment, to give a more reasonable estimate of worth. The formulas for the cost of equity, cost of debt, and free cash flow (relevant in the next section as well) can be searched in any accounting reference, and are not the focus of this set of notes.

3.2 DCF

Now that we understand WACC, we can define the Discounted Cash Flow. As we have seen above, the FV of a certain amount of money can be given by $(1+r)^t$. What if we want to go in the opposite direction? Taking a future value, and figuring out its present value is can be done just by a rearrangement of terms.

$$
\begin{aligned}
FV &= P(1+r)^t \\
P &= \frac{FV}{(1+r)^t}
\end{aligned}
$$

In this case, we refer to r not as the interest rate, but the discount rate, as it represents the amount be which we discount the future value when trailing back to the present. Now let us set up some notation. The future value we are interested in is the Free Cash Flow at some time t years into the future. Let's call this FCF_t. Next, we will set r equal to the WACC, which we will write as w. (The reason that we use WACC as the discount rate is that it represents the "opportunity cost" of an investment - if we are looking at the future value of an investment, we should discount it by the opportunity cost of sticking with that investment until the future. It is a natural price tag that allows us to compare different opportunities; you can ask yourself the question "How much return do I need to accrue in order to make this investment worth it?") Lastly, we will call the present value of a free cash flow in the future a **discounted cash flow**, and label it as $DCF_t^{(p)}$. This means that we take the free cash flow from year t and discount it down to year p. Using the formula above we have:

$$DCF_t^{(p)} = \frac{FCT_t}{(1+w)^{t-p}}$$

Notice the $t-p$ term at the bottom. Because we are discounting from year t to year p, we only need to discount by their difference. (Assume $t > p$.) If we are discounting all the way to the present, the formula naturally becomes:

$$DCF_t^{(0)} = \frac{FCT_t}{(1+w)^{t}}$$

Finally, consider the case of the free cash flow growing at a constant rate over the lifetime of a cash amount. That is, at year p we have FCF_p and that cash flow increases by a growth rate of g every year. Note that this is NOT the same thing as interest - it does not mean we have one cash amount that is growing in value, rather the cash flow produced by the company is $(1+g)$ times more every year, because the company itself is growing. Then we can write the free cash flow at year t as:

$$FCF_t = FCF_p \cdot (1+g)^{t-p}$$

And the discounted cash flow at year p as:

$$\begin{aligned} DCF_t^{(p)} &= \frac{FCT_t}{(1+w)^{t-p}} = \frac{FCF_p(1+g)^{t-p}}{(1+w)^{t-p}} \\ &= FCF_p \left(\frac{1+g}{1+w}\right)^{t-p} \\ DCF_t^{(0)} &= FCF_0 \left(\frac{1+g}{1+w}\right)^{t} \end{aligned}$$

4 A Model for Enterprise Value

We have come across many definitions and sets of notation in the previous sections, and it may be still a little unclear how everything fits together. Our last definition will be that of **enterprise value** or EV - the true measure of a company's price in the market. It is defined as:

$$EV = \text{Market Cap.} + \text{Total Debt - Total Cash}$$

While adding debt and subtracting cash might seem counterintuitive at first, think of EV less as value and more as the intrinsic price. When you buy a company, you also have to pay its debts, which would add to the price. When the company comes with cash, the price that you payed gets offset by that amount. If we know the EV of a company, we can subtract its debt, add its cash, and we would have a Market Capitalization.

$$\begin{aligned}\text{Market Cap.} &= EV - \text{Total Debt} + \text{Total Cash}\\ \text{Market Cap.} &= \text{Stock Price} \cdot \text{Num. Shares Outstanding}\\ \text{Stock Price} &= \frac{EV - \text{Total Debt} + \text{Total Cash}}{\text{Num. Shares Outstanding}}\end{aligned}$$

If we know the enterprise value, we can assign a value to the stock of a company! This is one of many ways to price a stock, and like we discussed before, in some sense, the average of all the valuations of a stock made by different analysts determine the stock price that you see presented online. How do we obtain the EV? The central theory of the discounted cash flow model is written below - that the sum of the present values of every free cash flow produced by a company from the present to the end of time, determines the EV.

$$EV = \sum_{t=0}^{\infty} DCF_t^{(0)}$$

Using this theory, we can price a company that we are hoping to analyze. Now, we dive more into the details - in practice, the procedure is slightly more complicated. First, we split this sum into two parts.

$$EV = \sum_{t=0}^{p} DCF_t^{(0)} + \sum_{t=p+1}^{\infty} DCF_t^{(0)}$$

The period from year 0 to year p is known as the **forecast period** of the model. In this period, we make projections and estimates of the free cash flow at each of the years in the period, and then discount them individually down to the present year. The reason we do this is become we believe we have some better idea of how the company may perform within the next few years (the time its current initiatives my start, end, etc.) and wish to use these values manually. However, we can only do this for a finite number of years, and our accuracy in predicting the free cash flow decreases with each year. For this reason, the forecast period is usually set to 10 years, or $p = 10$. The more interesting problem is how to handle the second term - the sum from the first year out of the forecast period to $t = \infty$.

Recall that an infinite geometric series has the form $\sum_{t=0}^{\infty} ar^t$, where a and r are a constants, with $|r| < 1$. The sum of this series is equal to $\frac{a}{1-r}$. Notice that this is the case of the sum index starting at 0. When the series is summed from an arbitrary integer m, the sum is given by:

$$\sum_{t=m}^{\infty} ar^t = \frac{ar^m}{1-r}$$

We revisit the second term of the enterprise value using this tool we have. We assume a constant growth rate g after the forecast period ends, that is the free cash flow at year p grows by $(1 + g)$

times every year. Now we can model the DCF as a term of this geometric series (see formula in previous section). To make things easier, let us define the index of the sum to be n, the number of years after the end of the forecast period that we are currently in. This is the same as saying $t = n + p$ for $t > p$.

$$\begin{aligned}\sum_{t=p+1}^{\infty} DCF_t^{(0)} &= \sum_{n=1}^{\infty} DCF_{n+p}^{(0)} \\ &= (1+w)^{-p} \sum_{n=1}^{\infty} DCF_{n+p}^{(p)}\end{aligned}$$

Here, we discount the DCF down to year p within the sum, and then outside of the sum we discount it from year p to year 0.

$$\begin{aligned}(1+w)^{-p} \sum_{n=1}^{\infty} DCF_{n+p}^{(p)} &= (1+w)^{-p} \sum_{n=1}^{\infty} FCF_p \left(\frac{1+g}{1+w}\right)^n \\ &= (1+w)^{-p} \cdot FCF_p \cdot \sum_{n=1}^{\infty} \left(\frac{1+g}{1+w}\right)^n \\ a &= 1 \\ r &= \left(\frac{1+g}{1+w}\right) \\ m &= 1\end{aligned}$$

We evaluate the sum to be:

$$\begin{aligned}\sum_{n=1}^{\infty} \left(\frac{1+g}{1+w}\right)^n &= \frac{(1+g)(1+w)^{-1}}{1 - \frac{1+g}{1+w}} \\ &= \frac{1+g}{1+w-(1+g)} \\ &= \frac{1+g}{w-g}\end{aligned}$$

Note one last detail about the sum - a geometric series can only converge of the common ratio r is less than 1 in absolute value. This means that $|\frac{1+g}{1+w}| = \frac{1+g}{1+w} < 1$. This is only possible when the WACC is greater than the growth rate. Similarly, in the TV term, we have $w - g$ on the bottom, and having the growth rate be greater yields a negative term for this valuation, which would not make sense. In both cases, it is clear to see that for a DCF analysis to be applicable, the WACC must be greater than the predicted growth rate (the growth rate is also a decision somewhat made by us, so this can be controlled). Putting it all together, we have:

$$\begin{aligned}\sum_{t=p+1}^{\infty} DCF_t^{(0)} &= (1+w)^{-p} \cdot FCF_p \cdot \frac{1+g}{w-g} \\ &= \frac{1}{(1+w)^p} \cdot \left(\frac{FCF_p(1+g)}{w-g}\right)\end{aligned}$$

The term $\left(\frac{FCF_p(1+g)}{w-g}\right)$ is usually called the **terminal value**, as it is the sum of the discounted cash flows from year $p+1$ to infinity, in year p dollars. This is often denoted as TV. Like we mentioned earlier, the discounted cash flows in the forecast period are calculated manually, by predicting the FCF at every year then discounting it. For the last term, we add some assumptions into the model (constant growth rate) to use the infinite series sum. Our new formula for the enterprise value becomes:

$$\begin{aligned} EV &= \sum_{t=0}^{p} DCF_t^{(0)} + \frac{TV}{(1+w)^p} \\ &= \sum_{t=0}^{p} \frac{FCF_t}{(1+w)^t} + \frac{TV}{(1+w)^p} \end{aligned}$$

Using this method, we can price companies. In practice, there are more intricacies still, especially in the forecasting of free cash flow. Interview questions about the DCF model can range from qualitative understanding of the time value of money, to being asked to evaluate a company in the interview itself. There is both mathematical theory and practical artistry to using this method successfully for valuation.